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Unique Determination of Solutions to the Burnett Equations

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Recent success in applying the Burnett equations to the one-dimensional shock-structure problem has raised the issue of whether the full Burnett equations can be used to replace the Navier-Stokes equations for solving boundary-value problems in rarefied gasdynamics. As is familiar from the classical rarefied gasdynamics literature, the Burnett equations, if not solved as a successive approximation to the Navier-Stokes equations for a small Knudsen number, would require more boundary conditions than those in the Navier-Stokes system, owing to the presence of the higher-order derivatives. In this paper, this issue is examined with concrete solution examples for the steady Couette flows, addressing specifically whether solutions to the full Burnett equations can be uniquely determined without adding more boundary conditions than those in the Navier-Stokes system. The analysis, supported by detailed numerical solutions, confirms that additional boundary conditions are needed as long as the Knudsen number is not identically zero, lest the solution to the Burnett equations is not unique.

Introduction

FOR the problem of one-dimensional shock structure, the existence of a solution to the Burnett equations[1] has been studied in Refs. 2-4. Montgomery[3] proved the existence of the Burnett solution of a weak plane shock for the case of an upstream Mach number asymptotically close to unity. For the one-dimensional shock-structure problem, the existence of the solution to the Burnett equations for any arbitrary upstream Mach number still remains unresolved.[2-4] Holway[4] gave an upper limit on the Mach number ($M \leq 1.851$) for the Grad[5] and Chapman-Enskog[6] methods based on a criterion for the convergence of their expansion procedures. It may also be shown that the shock-structure solution of the Burnett equations has a peculiar oscillatory behavior on the upstream side at $M > 1.478$ and is considered "nonphysical oscillation" by Bobylev.[7] The ill-posedness of the Burnett equations was also noted by Bobylev,[7] who showed that their solution exhibits a temporal instability when the Knudsen number, based on wavelength, exceeds the critical value of 2.5. In deriving the equations for the constitutive relations, Burnett[1] eliminated certain terms involving convective derivatives by using inviscid/isentropic relations. Consequently, the resulting Navier-Stokes and Burnett terms can represent only small corrections/perturbations to the inviscid Euler system. This assumption cannot be (uniformly) valid in a boundary/shear layer, where viscous and heat transfer terms must rank equally with the (Euler) convective effect in the differential equations expressing the conservation laws.[8]

As commonly observed in the literature on rarefied gasdynamics, the higher-order derivatives present in the Burnett equations would require more boundary conditions than those in the Navier-Stokes system.[9] The solution uniqueness and the proper boundary conditions of the Burnett equations remain as unresolved issues to date. This difficulty could disappear if one were to treat the Burnett terms as small perturbations from Navier-Stokes equations in a formal expansion procedure as in Burnett's original development. This has been the viewpoint shared by Schamberg[10] and Yang.[11]

Besides the formal expansion procedure, one could adopt the procedure proposed by Makashev[12] to avoid the issue on uniqueness and proper boundary conditions as mentioned earlier. Utilizing the first-order Navier-Stokes solution into Burnett's terms and into Schamberg's higher-order slip-wall conditions, Makashev[12] generated additional boundary conditions. Since the additional boundary conditions are generated with the help of the first-order Navier-Stokes solution, the procedure proposed by Makashev does not depart from Schamberg and Yang's perturbation procedure fundamentally and can be viewed as a composite/hybrid approach.

In spite of the unresolved issue on the solution existence of the Burnett equations,[2-4] a recent numerical study[13] shows that the Burnett solution can significantly improve Navier-Stokes solutions for the one-dimensional shock-structure problem. Encouraged by this study, researchers have made attempts[14-16] to apply the full Burnett equations to the boundary-value problem of rarefied hypersonic flows. As mentioned, the higher-order derivatives present in the Burnett equations should require more boundary conditions than those in the Navier-Stokes system.[9] The numerical solutions to the Burnett equations with Navier-Stokes type boundary conditions are obtained by Zhong et al.[14-16] without addressing, however, the issue of the solution uniqueness. (The issues of extra boundary conditions and the solution uniqueness were noted briefly in Ref. 16.) On the other hand, the equation system in question could possess an unsuspected structural redundancy to permit such a degeneracy. Thus, the proper number of boundary conditions in the problem formulation becomes an important issue, as long as the Burnett equations are treated and solved as a system of full equations (treating all terms as being equally important, as in Refs. 14-16). It is worthwhile to note that the Burnett equations to the one-dimensional shock-structure problem do not exhibit this particular difficulty, since upstream and downstream of the shock structure are the equilibrium states, given by Rankine-Hugoniot conditions.

In the present study, the issue is examined for the case of a steady Couette flow with supporting solution examples; the study confirms that more boundary conditions than in the Navier-Stokes system are required for a unique determination of the Burnett solution. Specifically, two more degrees of freedom remain to be specified/fixed by the additional boundary conditions. Through these extra boundary (wall) conditions, families of solutions satisfying the (same set of) Navier-Stokes boundary conditions can be obtained. Whereas a certain special iterative procedure applied to the Burnett equations, utilizing extrapolation data from the interior, may appear to succeed in bypassing this difficulty, the following will demonstrate that nonuniqueness still can arise even in such a procedure, since the results in this case are found to be strongly dependent on the initial guess/data in the iterative solution procedures.

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Steady Couette Flow

The problem of the Couette flow generated by steady relative motion of two parallel plates is considered as an example to study the issue on boundary conditions in question (refer to Fig. 1). The variables in the equations governing the Couette flow are nondimensionalized as follows:

$$y=\frac{\bar{y}}{H},\quad u=\frac{\bar{u}}{\sqrt{R\bar{T}_*}},\quad T=\frac{\bar{T}}{\bar{T}_*},\quad \rho=\frac{\bar{\rho}}{\bar{\rho}_*}$$

$$p=\frac{\bar{p}}{\bar{p}_*},\quad p_{ij}=\frac{\bar{p}_{ij}}{\bar{\rho}_*R\bar{T}_*},\quad q_i=\frac{\bar{q}_i}{\bar{\rho}_*(R\bar{T}_*)^{3/2}},\quad \mu=\frac{\bar{\mu}}{\bar{\mu}_*}$$

where H is the width between two parallel plates, $R=C_p-C_v$ with C_p and C_v being the specific heats at constant pressure and volume, respectively; $\bar{\rho}_*$, $\bar{p}_*$, $\bar{T}_*$, and $\bar{\mu}_*$ represent the density, thermodynamic pressure, temperature of the reference state and the dynamic viscosity of the fluid based on $\bar{T}_*$, respectively; and $\bar{p}_{ij}$ and $\bar{q}_i$ are the deviatorial pressure tensor and heat flux, respectively. The conservation equations are

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{bmatrix} p_{12} \\ p+p_{22} \\ up_{12}+q_2 \end{bmatrix}=0$$

For the Couette flow problem, the constitutive relations given originally by Burnett[1] and corrected by Wang-Chang and Uhlenbeck[17] are

$$p_{12}=-Kn\,\mu\frac{\mathrm{d}u}{\mathrm{d}y}$$

$$q_2=-Kn\frac{\gamma}{Pr(\gamma-1)}\mu\frac{\mathrm{d}T}{\mathrm{d}y}$$

$$\begin{aligned} p_{22}=Kn^2\frac{\mu^2}{p}\Bigg[&\alpha_9\frac{T}{p}\frac{\mathrm{d}^2p}{\mathrm{d}y^2}+\alpha_{11}\frac{T}{p^2}\left(\frac{\mathrm{d}p}{\mathrm{d}y}\right)^2 \\ &+(\alpha_{12}-2\alpha_9-2\alpha_{11})\frac{1}{p}\frac{\mathrm{d}T}{\mathrm{d}y}\frac{\mathrm{d}p}{\mathrm{d}y} \\ &+(\alpha_{13}+2\alpha_9+\alpha_{11}-\alpha_{12})\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}y}\right)^2 \\ &+(\alpha_7-\alpha_9)\frac{\mathrm{d}^2T}{\mathrm{d}y^2}+\alpha_6\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\Bigg] \end{aligned}$$

where $\gamma\equiv C_p/C_v$ is the specific heat ratio, and Kn is the reduced Knudsen number defined as

$$Kn=\frac{\bar{\mu}_*}{\bar{\rho}_*\sqrt{R\bar{T}_*}H}$$

and can be related to the Knudsen number in common use $Kn_*\equiv\bar{\lambda}_*/H$, as $Kn=0.78\,Kn_*$, where $\bar{\lambda}_*$ is the mean free path of the reference state (note that $\bar{\lambda}_*=16\bar{\mu}_*/5\bar{\rho}_*\sqrt{2\pi R\bar{T}_*}$).

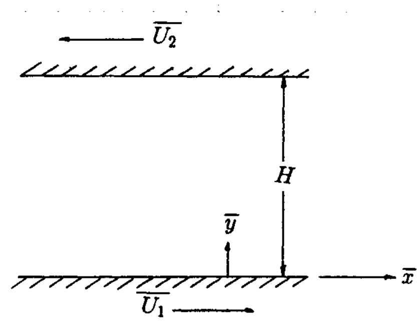

Fig. 1 Illustration of Couette flow coordinate system.

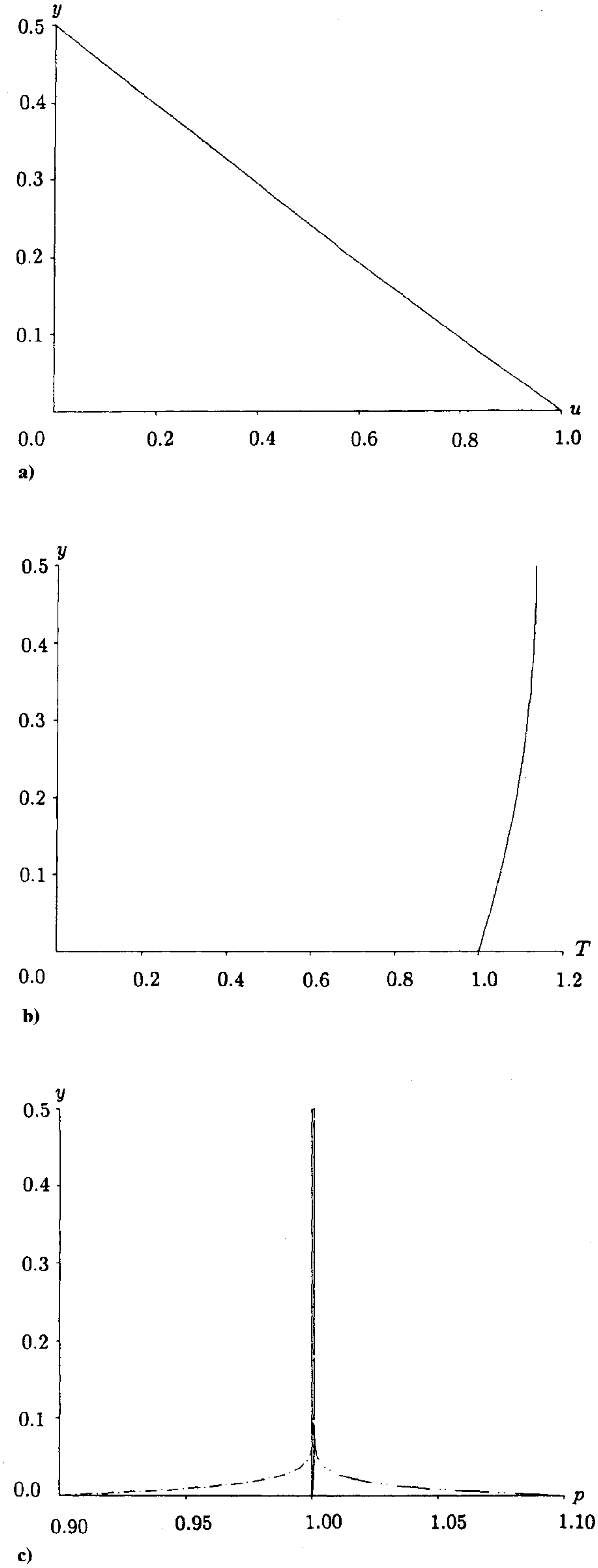

Fig. 2 Nonslip-wall solutions for $U_1=1.0$, $U_2=-1.0$, $T_1=T_2=1.0$, $P_0=1.0$, and $Kn=0.01$: a) velocity u, b) thermodynamic temperature T, and c) thermodynamic pressure p as a function of y. ——, case 1 (Navier-Stokes); —·—, case 2 (Burnett, $P_b=0.9$); ----, case 3 (Burnett, $P_b=1.0$); —··—, case 4 (Burnett, $P_b=1.1$); ———, case 5 (Burnett, extrapolation). Note that the solutions u and T of the Navier-Stokes and Burnett equations are identical for nonslip-wall conditions, and the solution p of case 3 coincides with that of case 5. The wall-layer structures can be seen in the p-profile plot.

The constants α are $\alpha_6 = -0.667$, $\alpha_7 = 0.667$, $\alpha_9 = -1.333$, $\alpha_{11} = 1.333$, $\alpha_{12} = -1.333$, and $\alpha_{13} = 2.0$ for Maxwell gas.[13,14,16] With the constitutive relations defined above, the conservation equations can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\mu\frac{\mathrm{d}u}{\mathrm{d}y}\right) = 0 \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[u\mu\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\gamma}{Pr(\gamma-1)}\mu\frac{\mathrm{d}T}{\mathrm{d}y}\right] = 0 \tag{2}$$

$$\begin{aligned} p + Kn^2\frac{\mu^2}{p}\bigg[&\alpha_9\frac{T}{p}\frac{\mathrm{d}^2p}{\mathrm{d}y^2} + \alpha_{11}\frac{T}{p^2}\left(\frac{\mathrm{d}p}{\mathrm{d}y}\right)^2 \\ &+ (\alpha_{12} - 2\alpha_9 - 2\alpha_{11})\frac{\mathrm{d}T}{\mathrm{d}y}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}y} \\ &+ (\alpha_{13} + 2\alpha_9 + \alpha_{11} - \alpha_{12})\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}y}\right)^2 \\ &+ (\alpha_7 - \alpha_9)\frac{\mathrm{d}^2T}{\mathrm{d}y^2} + \alpha_6\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\bigg] = P_0 \end{aligned} \tag{3}$$

where P_0 is a constant resulting from an integration of the normal momentum equation. For a nonslip wall, the fluid next to the wall moves with the plate, and the temperature takes on the prescribed wall temperature. For walls allowing velocity slip and temperature jump, the classical slip-wall conditions[5,10,18] are used, namely,

$$u = U_{\text{wall}} + \frac{2-\sigma_u}{\sigma_u}\frac{16}{5\sqrt{2\pi}}Kn\frac{\sqrt{T}\mu}{p}\frac{\mathrm{d}u}{\mathrm{d}N} \tag{4}$$

$$T = T_{\text{wall}} + \frac{2-\sigma_T}{\sigma_T}\frac{2\gamma}{Pr(\gamma+1)}\frac{16}{5\sqrt{2\pi}}Kn\frac{\sqrt{T}\mu}{p}\frac{\mathrm{d}T}{\mathrm{d}N} \tag{5}$$

where N is the outward normal of the wall, and σ_u and σ_T are the accommodation coefficients, which are adequate if the slip/jump is not excessively large.

As can be shown easily, Eq. (3) is reduced simply to $p = P_0$, and Eqs. (1) and (2) are unchanged for the Navier-Stokes system. For a nonslip wall, equations for u and T are decoupled from that of p, and the Navier-Stokes and Burnett systems share the same solutions of u and T. The reduced equations, Eqs. (1-3), indicate that seven boundary conditions are needed for a unique determination of the solutions. More specifically, in addition to the Navier-Stokes type boundary conditions (two for u, two for T, and $p = P_0$), the Burnett equations have two more integration constants to be specified, which represents two degrees of freedom.

Zhong et al.[14,16] introduced additional terms to constitutive relations to treat instability of the Burnett equations. The "augmented Burnett equations" in this case become

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\mu\frac{\mathrm{d}u}{\mathrm{d}y} - Kn^2\frac{3\omega_7}{4}\frac{\mu^3}{p^2}T\frac{\mathrm{d}^3u}{\mathrm{d}y^3}\right) = 0 \tag{6}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}\bigg[&u\mu\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\gamma}{Pr(\gamma-1)}\mu\frac{\mathrm{d}T}{\mathrm{d}y} - Kn^2\frac{3\omega_7}{4}\frac{\mu^3}{p^2}Tu\frac{\mathrm{d}^3u}{\mathrm{d}y^3} \\ &- Kn^2\frac{\mu^3}{p\rho}\left(\theta_7\frac{\mathrm{d}^3T}{\mathrm{d}y^3} + \theta_6\frac{T}{\rho}\frac{\mathrm{d}^3\rho}{\mathrm{d}y^3}\right)\bigg] = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} p + Kn^2\frac{\mu^2}{p}\bigg[&\alpha_9\frac{T}{p}\frac{\mathrm{d}^2p}{\mathrm{d}y^2} + \alpha_{11}\frac{T}{p^2}\left(\frac{\mathrm{d}p}{\mathrm{d}y}\right)^2 \\ &+ (\alpha_{12} - 2\alpha_9 - 2\alpha_{11})\frac{\mathrm{d}T}{\mathrm{d}y}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}y} \\ &+ (\alpha_{13} + 2\alpha_9 + \alpha_{11} - \alpha_{12})\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}y}\right)^2 \\ &+ (\alpha_7 - \alpha_9)\frac{\mathrm{d}^2T}{\mathrm{d}y^2} + \alpha_6\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\bigg] = P_0 \end{aligned} \tag{8}$$

which would require eleven boundary conditions for the unique determination of the solution (four for u, four for T, and three for p). Three additional coefficients—ω_7, θ_6, and θ_7—were so selected that the augmented Burnett equations become linearly stable; for Maxwell gas, they are $\omega_7 = 2/9$, $\theta_6 = -5/8$, and $\theta_7 = 11/16$.[14,16] As already noted, one could avoid the issue by treating the Burnett terms as well as the augmented Burnett terms as small perturbations to the Navier-Stokes equations in a formal expansion for an asymptotically small Kn.[10,11] If, on the other hand, the full Burnett or augmented Burnett equations are solved as in Refs. 14-16, the solutions cannot be uniquely determined without specifying additional boundary conditions. The following computational study will demonstrate that, by providing more boundary conditions in addition to those of Eqs. (4) and (5), different solutions to the Burnett equations can be obtained.

The nonconservative form of Eqs. (1-3), with either nonslip-wall or slip-wall boundary conditions, are solved iteratively and discretized by finite difference in delta form; for example, Eq. (1) can be discretized as

$$\begin{aligned} &\delta_{yy}\Delta u + \left(\frac{\omega}{T}\frac{\mathrm{d}T}{\mathrm{d}y}\right)_n\delta_y\Delta u + \left(\frac{\omega}{T}\frac{\mathrm{d}u}{\mathrm{d}y}\right)_n\delta_y\Delta T - \left(\frac{\omega}{T^2}\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}T}{\mathrm{d}y}\right)_n\Delta T \\ &\quad = -\left(\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \frac{\omega}{T}\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}T}{\mathrm{d}y}\right)_n \end{aligned}$$

where ω is the exponent in the viscosity-temperature law ($\mu = T^\omega$), $\Delta u \equiv u_{n+1} - u_n$, the subscript n denotes the nth iteration, and δ_{yy} and δ_y represent the second-order accurate central finite difference form of the second and the first y derivatives, respectively. The reduced coupled finite difference equations are solved by band-matrix solver from LINPACK.

Discussion of Specific Examples

For simplicity, only the Maxwell gas is considered, for which $\gamma = 5/3$, $Pr = 2/3$, and $\omega = 1.0$. Without loss of generality, we let the lower plate move with velocity $U_1 = 1$ and the upper plate move with velocity $U_2 = -1$, and we choose the integration constant P_0 to be unity, i.e., $P_0 = 1.0$. For the solution uniqueness of the Burnett equations, the pressure at the walls must accordingly be prescribed. Again, for simplicity, we assume a symmetry in the boundary values of T and p, i.e., $T_1 = T_2 = T_b$ and $p_1 = p_2 = P_b$. The subscripts 1 and 2 refer to the lower and upper plates, respectively. This leaves one degree of freedom in the boundary conditions, which is the wall pressure P_b. Three wall-pressure values are considered in the following study, namely $P_b = 0.9$, 1.0, and 1.1. The solutions to be examined in Figs. 2-5 will be classified and labeled for five cases depending on the governing equations and how the extra degree of freedom is handled: case 1 is the Navier-Stokes solution (solid curves), cases 2-4 represent the solutions of the Burnett equations obtained by prescribing $P_b = 0.9$ (solid-dot curves), 1.0 (dash curves), and 1.1 (solid-dot-dot curves), respectively; case 5 is a Burnett solution for which a procedure employing extrapolation from the interior (as in Refs. 14-16) is adopted, using either $p = 1.0$ (everywhere) or the Navier-Stokes solution as an initial guess. Three reduced Knudsen numbers are considered in this study, namely, $Kn = 0.01$, 0.1, and 1.0. Results shown in Figs. 2-5 are converged solutions with residue less than 10^{-5}. With the boundary conditions mentioned earlier, T and p are symmetric and u is antisymmetric with respect to $y = 0.5$ (confirmed by numerical solutions); hence, only half of the solution range is shown for each profile. Results obtained for nonslip-wall conditions are presented for $Kn = 0.01$ in Figs. 2a-2c and for

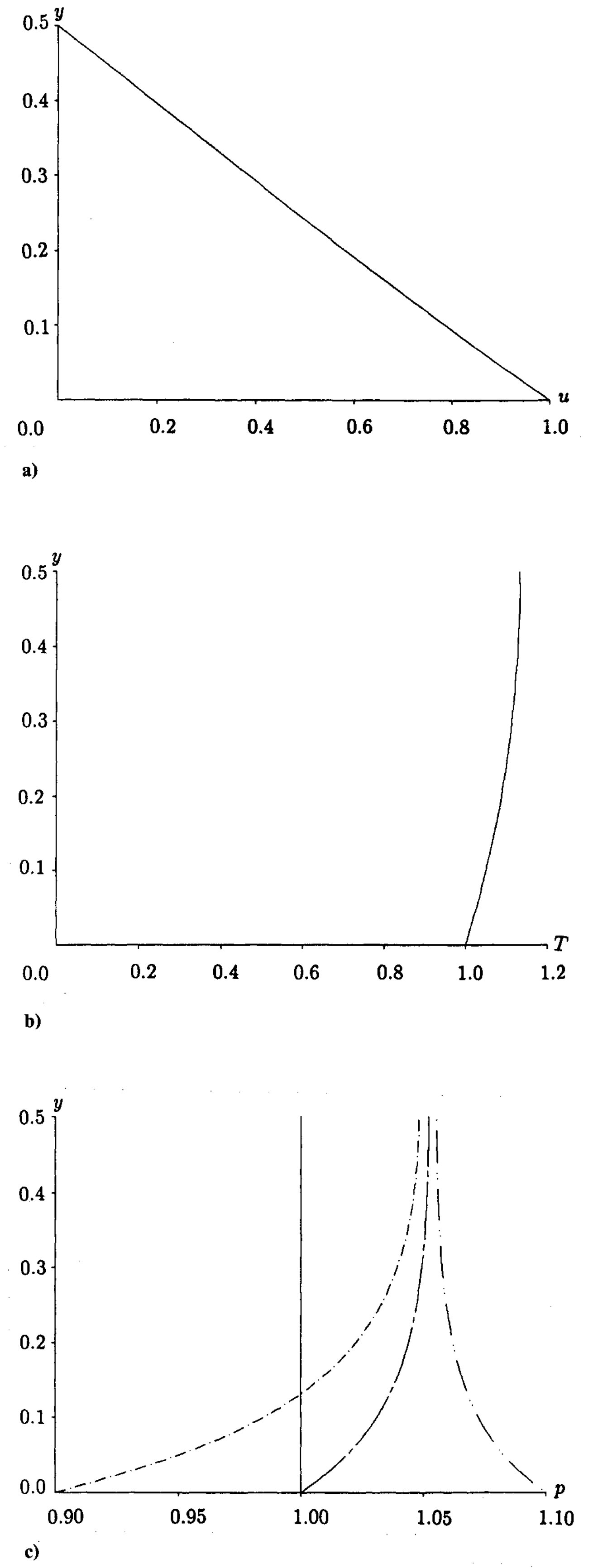

Fig. 3 Nonslip-wall solutions for $U_1 = 1.0$, $U_2 = -1.0$, $T_1 = T_2 = 1.0$, $P_0 = 1.0$, and $Kn = 0.1$: a) velocity u, b) thermodynamic temperature T, and c) thermodynamic pressure p as a function of y. ——, case 1 (Navier-Stokes); — · —, case 2 (Burnett, $P_b = 0.9$); ----, case 3 (Burnett, $P_b = 1.0$); — · · —, case 4 (Burnett, $P_b = 1.1$); ———, case 5 (Burnett, extrapolation). As mentioned, the solution p of case 3 coincides with that of case 5.

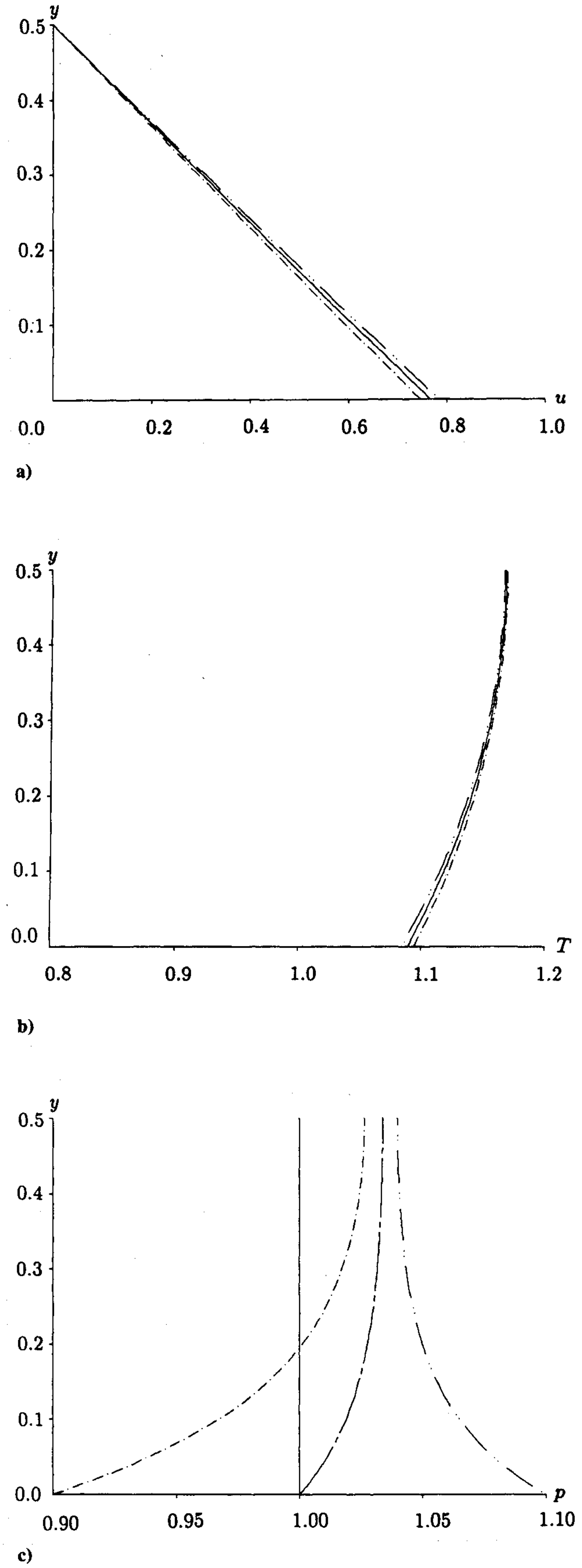

Fig. 4 Slip-wall solutions for $U_1 = 1.0$, $U_2 = -1.0$, $T_1 = T_2 = 1.0$, $P_0 = 1.0$, and $Kn = 0.1$: a) velocity u, b) thermodynamic temperature T, and c) thermodynamic pressure p as a function of y. ——, case 1 (Navier-Stokes); — · —, case 2 (Burnett, $P_b = 0.9$); ----, case 3 (Burnett, $P_b = 1.0$); — · · —, case 4 (Burnett, $P_b = 1.1$); ———, case 5 (Burnett, extrapolation). Note that the solutions u and T of case 1 coincide with those of cases 3 and 5.

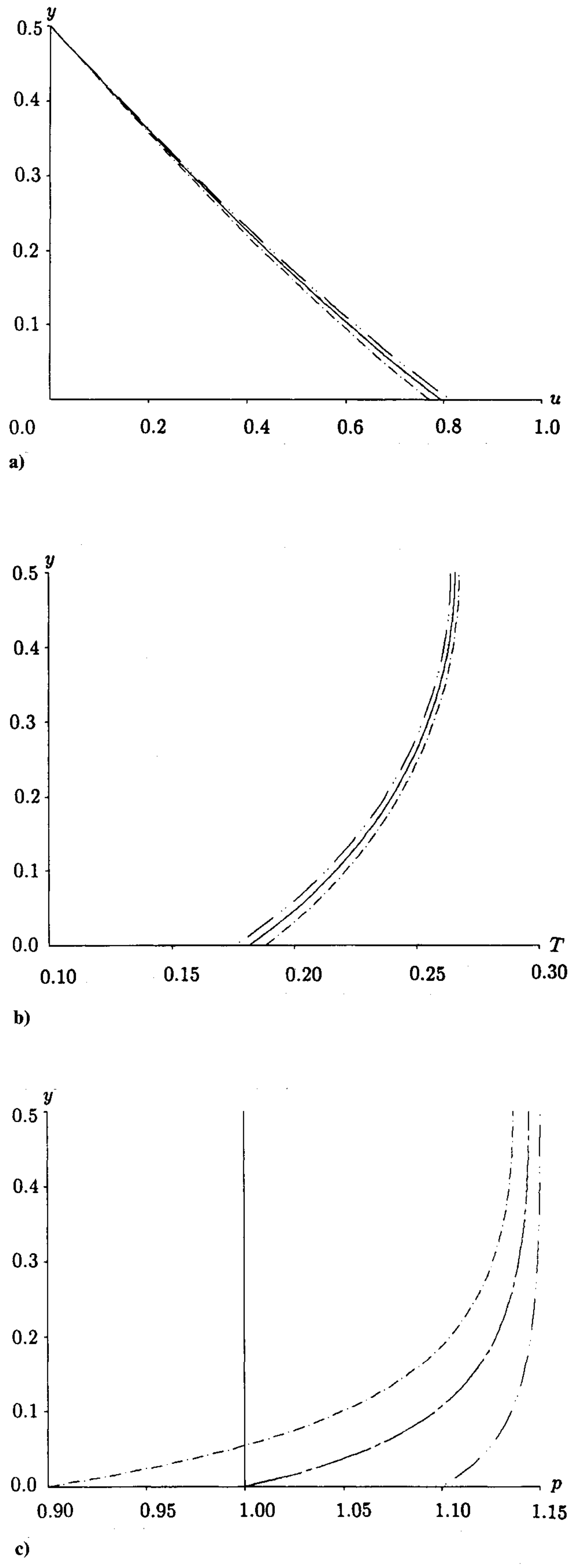

Fig. 5 Slip-wall solutions for $U_1 = 1.0$, $U_2 = -1.0$, $T_1 = T_2 = 1.0$, $P_0 = 0.1$, and $Kn = 1.0$: a) velocity u, b) thermodynamic temperature T, and c) thermodynamic pressure p as a function of y. ——, case 1 (Navier-Stokes); —·—, case 2 (Burnett, $P_b = 0.9$); ----, case 3 (Burnett, $P_b = 1.0$); —··—, case 4 (Burnett, $P_b = 1.1$); ———, case 5 (Burnett, extrapolation).

$Kn = 0.1$ in Figs. 3a-3c. Solutions obtained from slip-wall conditions are presented for $Kn = 0.1$ in Figs. 4a-4c and for $Kn = 1.0$ in Figs. 5a-5c.

Of the solutions for nonslip-wall conditions at $Kn = 0.01$ shown in Figs. 2a-2c, the temperature is set at $T_1 = T_2 = T_b = 1.0$. The u and T are decoupled from p in this example and are plotted as a function of y in Figs. 2a and 2b, respectively. They are shared by all five cases considered. The resulting pressure profiles are given in Fig. 2c, in which the Burnett solution for p in case 3 (with the p at the wall equal to unity) coincides with the pressure profile from the Burnett solution of case 5 (which uses an extrapolation procedure) to within the solution accuracy. As it turns out, the coincidence of the solutions for p of cases 3 and 5 occurs also in Figs. 3-5, irrespective of Kn and wall-slip effects. Except near the walls, the solutions of the Burnett equations, with different types of boundary conditions, agree quite well with Navier-Stokes solutions, owing to the small Knudsen number in these examples. As expected, there are wall layers near both solid plates with thickness estimated to be of order Kn, because the highest derivative term is multiplied by Kn^2, and these wall layers will not vanish as long as Kn is not identically zero.

The next example shown in Fig. 3 is again for nonslip conditions at the wall with $T_b = 1.0$ but a larger reduced Knudsen number, $Kn = 0.1$. The profiles of velocity u, temperature T, and the corresponding pressure p are presented in Figs. 3a, 3b, and 3c, respectively. The last figure of $p(y)$ indicates that there is a substantial difference between the Navier-Stokes and Burnett calculations, as well as among Burnett solutions with different types of boundary conditions.

It is essential to note that the recovery of the case 3 solution with $p_1 = p_2 = 1$ from the case 5 solution based on an extrapolation procedure does not mean that the boundary condition $p_1 = p_2 = 1$ is an inherent property of the extrapolation procedure. Rather, it is conditioned/affected by the initial data used in the iteration. Solutions of cases 2 and 4 with $p_1 = p_2 \neq 1$ can, and have been, reproduced (with the residue two orders smaller than those in cases 2 and 4) with the extrapolation procedure (used in case 5) by using these solutions as initial data for iterations. In fact, one should be able to recover these solutions (cases 2 and 4) with other procedures (for example, applying Eq. (3) at the wall to calculate the wall pressure) by using case 2 and 4 solutions as initial data, as long as the numerical procedure is accurate.

The need of additional boundary conditions for the Burnett equations and the dependence of the Burnett solutions (by the extrapolation method) on the initial data are again confirmed by the examples presented in Figs. 4a-4c for $Kn = 0.1$ and Figs. 5a-5c for $Kn = 1.0$, which allow velocity slip and temperature jump at the wall. Here both accommodation coefficients are taken to be unity, i.e., $\sigma_u = \sigma_T = 1$. Unlike the nonslip-wall examples, the Navier-Stokes and Burnett solutions for u and T do not coincide in general, but the Navier-Stokes result and Burnett solutions of cases 3 and 5 do agree. The solution for $Kn = 1.0$ in Figs. 5a-5c has a relatively low wall temperature, $T_1 = T_2 = 0.1$, which may simulate qualitatively the strong wall cooling characteristic of most hypersonic viscous flows of interest.

One could argue that the flow near the wall is near equilibrium for the problems considered in Refs. 14-16, i.e., the local Knudsen number is small, but as demonstrated by numerical examples given earlier that the nonuniqueness of the solution to the Burnett equations exists for any Knudsen number as long as Kn is not identically zero. From this view, the unfavorable result obtained earlier by Tannehill and Eisler,[19] which differs considerably from the corresponding direct simulation Monte Carlo calculation, may have been caused inadvertently by an unfavorable set of initial data rather than by the second-order slip-wall conditions (from Schamberg[10]) used in their calculation, since the final numerical solution is highly dependent on the initial data, and the additional boundary conditions were not provided.

Concluding Remarks

The foregoing analysis of the Couette flow problem shows that the Burnett equations cannot be uniquely determined with the same number and types of slip-wall or nonslip-wall conditions used in the Navier-Stokes system. As demonstrated by the examples, for the solutions to the Burnett equations to be uniquely determined, an additional boundary condition is needed on each wall, as long as $Kn \neq 0$, and different solutions will result from the choice of the free boundary values assigned to these extra boundary conditions. As indicated by the analysis, even if the solution procedure were restricted to the extrapolation strategy similar to that in Refs. 14–16, the solution cannot be uniquely determined, since the degrees of freedom still remain in the initial data set of the iterative procedure, which strongly affects the final solutions.

Acknowledgments

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